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c-Orderable Division Rings with Involution*

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DEDICATED TO THE MEMORY OF MY MOTHER.

Let R be any division ring with involution. The *$*$-core* (resp. *core*) of R is the set of elements of the form

$$c = p_1 + p_2 + \cdots + p_r,$$

where each term p_i is some non-zero product of norms aa^* (resp. squares a^2). T. Szele proved that in order for the division ring R to be Hilbert ordered (e.g., R has some total order relation, which is additive and multiplicative) it is necessary and sufficient that the core of R exclude 0. In this paper we shall investigate the division rings with involution R such that the $*$-core of R excludes 0. We call these division rings *c-orderable*. In fact, any c-orderable division ring R is shown to admit an ordering of the following type. For a, b, c, d arbitrary elements of R, we have

(1) $a \geqslant b$ implies $a^* \geqslant b^*$,

(2) $a \geqslant b$ and $c \geqslant d$ imply $a + c \geqslant b + d$,

(3) $a < b$ or $a = b$ or $a > b$ whenever $a = a^*$ and $b = b^*$, and

(4) $a \geqslant b$ implies $axx^* \geqslant bxx^*$.

Such an ordering we call a *c-ordering* (c for $*$-core). Among other things, we show that this ordering is conditionally Baer, that is,

(5) $s = s^* \geqslant 0$ implies $xsx^* \geqslant 0$, provided s can be bounded below and above by positive rationals.

Whether or not this proviso can be dropped is an open question. Another important property of the c-ordering is that the set of *bounded* elements x at this ordering (e.g., $xx^* \leqslant r_0$ for some rational r_0) is a *$*$-valuation subring* V

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in the sense given by Holland [6]. Here the residue division ring of V has dimension at most 4.

A well-known theorem of Albert asserts that every algebraic ordered division ring R is a field. Using essentially the same argument one can show the stronger result that every ordered division ring R is a purely transcendental extension of the center Z or R. One might ask whether the same conclusion will hold for R any c-ordered division ring, which is not quaternionic. Using new methods we show that if $x \in R$ is algebraic over some subfield F of central bounded elements, then the degree of x over Z is at most 2. When $x = x^*$ or if some central skew element $\sigma_0 \neq 0$ is algebraic over F, $x \in Z$ follows.

Conventions

1. In all that will follow the term "rational" unadorned will always refer to a positive rational in the usual sense (*natural* rational).

2. All our division rings R are such that 0 is not expressible as a sum of norms $xx^* \neq x$ (*naturally orderable* or **-formally real* division rings).

3. All our order relations are preserved under the involution, that is, $a \geqslant b$ implies $a^* \geqslant b^*$.

4. All order relations are additive: $a \geqslant b$ and $c \geqslant d$ imply $a + c \geqslant c + d$.

5. Non-zero norms xx^* are positive at all our order relations.

Central Definitions

Definition 1. (1) The *-core P of the ring with involution R is the set of sums

$$c = \sum_i p_i,$$

where each p_i is of the form $0 \neq p_i = aa^* \cdot bb^* \cdots tt^*$ $(a, b, \ldots, t \in R)$.

(2) If $0 \notin P$, we shall call R a *c-orderable* ring.

Definition 2. *Let $a \geqslant b$ be any ordering on the ring with involution R.*

(1) If $r_0 \leqslant xx^* \leqslant r_1$ for some (natural) rationals r_i, then x is called a unit.

(2) If $xx^* < r$ for each rational, then x is called an *infinitesimal.*

(3) If x is a unit or an infinitesimal, call x a *bounded element.*

DEFINITION 3 (Orderings). We say that the order relation $a \geqslant b$ is *induced* by the subset M if $a \geqslant b$ is equivalent with $a - b \in M \cup \{0\}$.

(1) For $M = \{\sum_i x_i x_i^* \mid 0 \neq x_i \in R\}$ the order relation induced by M is called the *natural ordering* (notation: $a \geqslant^{\text{nat.}} b$).

(2) For $M = P$ (= *-core of R), where R is c-orderable, the order relation induced by M is called the ordering of *natural type* (notation: $a >_1^{\text{nat.}} b$).

(3) *By c-ordering* we mean an order relation such that

(o) $1 > 0$,

(i) $a \geqslant b$ implies $a^* \geqslant b^*$,

(ii) $a \geqslant b$, $c \geqslant d$ imply $a + c \geqslant b + d$,

(iii) $s < d$, $s = d$, or $s > d$ for each $s = s^*$, $d = d^*$,

(iv) *$a \geqslant b$ implies $axx^* \geqslant bxx^*$.*

(4) By *Baer* ordering we mean an order relation such that (o)–(iii) are true and

(iv)′ $a \geqslant b$ implies $xax^* \geqslant xbx^*$.

(5) By *q-ordering* we mean a c-ordering such that the involution is the identity mapping.

Note. It can be easily checked that all orderings defined above obey our conventions. Also, the difference between a Baer ordering as defined an the original definition due to Holland (see [5]) is slight: If $\geqslant$ is a Baer ordering in our sense, then the order relation induced by $M^{\text{sym}} = \{x \mid x = x^* > 0\}$ is a Baer ordering in Holland's sense.

DEFINITION 4 (Comparability). Let $\geqslant$ be a fixed c-ordering on the division ring R, let a, b be positive elements, and let c be an element with a sign (e.g., $c > 0$, $c = 0$, or $c < 0$).

(1) Define $|c| = c$ if $c \geqslant 0$, and $|c| = -c$ if $c < 0$.

(2) Write $a \ll b$ if $a < rb$ for each rational r.

(3) Write $a \approx b$ if $a < (1 + r)b$ and $b > (1 + r)a$ for each rational r.

(4) Write $a \, O \, b$ if $a \not\ll b$ and $b \not\ll a$ (*archimedean equivalent pair*).

1 GENERAL FACTS

Recall that the *-core P of the division ring R is the set of arithmetical sums of products of norms $xx^* \neq 0$. Equivalently, P is the additive and multiplicative subset generated by all non-zero norms. Thus P is closed under sums and products, and $P=P^*$ contains all norms $xx^*\neq 0$. Is P preserved under translation $(a\to xax^*)$? Under conjugation? Since we pass from xax^* to xax^{-1} by right multiplication by the norm $(x^*)^{-1}x^{-1}$, we see that these questions are equivalent. Since conjugates of norms are products of two norms

$$xaa^*x^{-1}=((xa)(xa)^*)[(x^*)^{-1}x^{-1}],$$

and since conjugation is an automorphism, the questions have an affirmative answer. Also, P contains inverses, for we have

$$x^{-1}=x^*((x^{-1})^*x^{-1}).$$

We have shown the following theorem.

THEOREM 1. *The $*$-core P of any division ring R* (i) *is $*$-closed*, (ii) *is additive*, (iii) *is multiplicative*, (iv) *is preserved under translation* $(a\to xax^*)$, (v) *is preserved under conjugation and* (vi) *contains inverses.*

One can give the following equivalent definition of the c-ordering. There is some subset M such that

(C1) $M=M^*$,

(C2) $M+M\subseteq M$,

(C3) $M\cup(-M)\supseteq\{x\mid 0\neq x=x^*\}$,

(C4) $0\notin M$, $1\in M$, and $x\in M$ imply $xaa^*\in M$ for each $a\neq 0$, and

(C5) $a\geqslant b$ if and only if $a-b\in M\cup\{0\}$.

If R has some c-ordering it is clear that $xx^*\in M$ for each $0\neq x$. By (C4), $xx^*yy^*\in M$, for each $0\neq x, y$. By induction on n, $x_1x_1^*\cdots x_nx_n^*\in M$. Thus $P\subseteq M$ and $0\notin P$ follows. Conversely, suppose that R is any c-orderable division ring. Using Zorn's lemma we can find a maximal P-additive subset $M=M^*$ excluding 0. We proceed to show that M satisfies condition (C2) in the above. Let $0\neq s=s^*\in R$ and suppose that $-s\notin M$. Using properties (i)–(vi) of the $*$-core P appearing in Theorem 1, one can check quite easily that the subset of R

$$M_s=M\cup sP\cup(sP+M)$$

is, again, a P-additive subset excluding 0. By maximality of M, $M=M_s$ follows, that is, $s\in M$. We conclude that the binary relation defined by

$$a\geqslant b\Leftrightarrow a-b\in M\cup\{0\}$$

is a c-ordering on R. We have shown the basic

THEOREM 2. *In order for the division ring R to have some c-ordering it is necessary and sufficient that R be c-orderable.*

We proceed to

THEOREM 3. *Let R be any c-ordered division ring. We have*

(i) *for r any relative rational, $r > 0$ if and only if r is a (natural) rational;*

(ii) *for $a \in R$, $a > 0 \Leftrightarrow a^{-1} > 0$; and*

(iii) *for $a > b > 0$ and $c >_P d >_P 0$ (e.g., $d \in P$ and $c - d \in P$), we have $ac > bd$.*

Proof. (i) Since $P \subseteq M$, it follows that each natural rational r belongs to M. The converse is clear.

(ii) If $a > 0$ then $aa^{-1}(a^{-1})^* = (a^{-1})^* > 0$ follows, giving $a^{-1} > 0$.

(iii) Since $a > b$ and $d \in P$, it follows that $ad > bd$. Because $a > 0$ and $c - d \in P$, we get $ac > ad > bd$, and the theorem is proved.

Here are some additional facts.

Remark 1 (S. Berberian). For $a, b \in R$, we have

$$(a + b)(a + b)^* \leqslant 2(aa^* + bb^*).$$

Remark 2 (Holland). For $s = s^*$ and $k = -k^* \in R$, we have

$$|sk - ks| \leqslant s^2 - k^2.$$

Remark 3 (Vidav). For $s = s^*$ and r any rational, we have $-r \leqslant s \leqslant r$ if and only if $s^2 \leqslant r^2$.

Remark 4. For $s = s^*$ any symmetric such that $-r_0 \leqslant s \leqslant r_0 < 1$, some rational r_0, we have $s^2 < |s| + \varepsilon$ for each rational ε.

Remark 5. For s and k as in Remark 2, we have $|sk - ks| \ll s^2 - k^2$ provided s^2 and $-k^2$ provided s^2 and $-k^2$ are not archimedean equivalent.

Remark 6. Generally $|sk - ks| \ll s^2 - k^2$ if and only if $xx^* \approx x^*x$, where $x = s + k$ and $s = s^*$, $k = -k^* \in R$.

For Remark 1, use the identity

$$(a + b)(a + b)^* + (a - b)(a - b)^* = 2(aa^* + bb^*).$$

For Remark 2, use the identity

$$(s + k)(s + k)^* = (s^2 - k^2) - (sk - ks).$$

For Remark 3, use successively the identities

$$r-s=(1/2r)(r-s)^2+(1/2r)(r^2-s^2),$$
$$2r(r^2-s^2)=(r-s)(r+s)(r+s)^*+(r+s)(r-s)(r-s)^*.$$

For Remark 4, we separate two cases. If $|s|\leqslant r<1$ implies $|s|\leqslant r^2$ for each rational r, then $s\leqslant r_0<1$ implies $|s|\leqslant r_0^{2^n}$, $n=1,2,\ldots$. In this case, $|s|$ is then an infinitesimal. From Remark 3, s^2 is also an infinitesimal so that $s^2<\varepsilon\leqslant|s|+\varepsilon$, for each rational ε. If, on the other hand, there is some rational r_1 with $|s|\leqslant r_1$ but $|s|\nleqslant r_1^2$, then $|s|>r_1^2$ follows and, by Remark 3 $s^2\leqslant r_1^2<s$, so, again, $s^2<|s|+\varepsilon$.

For Remark 5, let us suppose, for instance, $s^2\lll -k^2$. Then $s^2\lll-(rk)^2$ for each rational r. By Remark 2, $r\,|sk-ks|=|s(rk)-(rk)s|\leqslant s^2-(rk)^2\leqslant-2r^2k^2$, so that $|sk-ks|\leqslant-2rk^2$. From this, $|sk-ks|\lll-k^2\leqslant s^2-k^2$, resulting in $|sk-ks|\lll s^2-k^2$.

The justification of Remark 6 and those to follow are left as exercises to the reader.

Remark 7. (1) The bounded elements (see Definition 3.3) form a $\mathbb{Q}$-additive subgroup ($\mathbb{Q}=$ relative rationals).

(2) The infinitesimals (see Definition 3.2) form a $\mathbb{Q}$-subgroup.

Remark 8. For $s=s^*$ any positive element and $x\in R$, we have

(1) s is bounded if and only if $s\leqslant r_0$ for some rational r_0;

(2) s is a unit if and only if $r_1\leqslant s\leqslant r_0$ for some rationals r_1, r_0;

(3) s is an infinitesimal if and only if $s<r$ for each rational; and

(4) x is bounded (resp. unit, infinitesimal) if and only if xx^* is bounded (resp. unit, infinitesimal).

6. The Bounded Elements

Let R be any c-ordered division ring. Denote by V (resp. J) its subset of bounded elements (resp. infinitesimals). We know from Remark 7 that both V and J are additive $\mathbb{Q}$-subgroups ($\mathbb{Q}=$ relative rationals). Actually, we can establish the following structure theorem.

Theorem 4. *Let R be any c-ordered division ring, let V be the set of bounded elements, and let J be the subset of infinitesimals.*

(1) *V is a *-closed valuation subring of R.*

(2) *The center Z_b of the ring V is a valuation domain of the field Z.*

(3) *The set on non-units of the ring V is precisely the infinitesimals J, thus, J is a $*$-ideal of the ring V.*

(4) *The residue division ring V/J has dimension at most 4. In fact, all symmetrics of V/J are central elements.*

Proof. (1) and (2). As noted in Remark 7, V and J are additive subgroups. We proceed to show that $x \in V$ implies $x^* \in V$. By construction $xx^* \leqslant r_0$, for some rational r_0. Now

$$1 + r_0 \geqslant 1 + xx^* \geqslant x + x^* \geqslant -(1 + xx^*) \geqslant -(1 + r_0),$$

for $1 + xx^* - (x + x^*) = (1 - x)(1 - x)^* \geqslant 0$ and $x + x^* + (1 + xx^*) = (1 + x)(1 + x)^* \geqslant 0$. By Remark 3, $(x + x^*)^2 \leqslant (1 + r_0)^2$, so that $x + x^* \in V$. Thus $x^* = (x + x^*) - x \in V$. For the multiplicative property of V, let us show firstly that $xs^4x^* \in V$, where $s = s^*$ and x are in V. Now $x^* \in V$ gives $r_0 \geqslant x^*x$ for some rational r_0. Then $r_0 s^2 + x^*xs^2$, $s^2 r_0 s^2 \geqslant s^2 x^* x s^2$. However, from $s \in V$ follows $r_1 \geqslant s^2 \geqslant 0$ for some rational r_1. By Remark 3, $r_1^2 r_0 \geqslant s^4 r_0 \geqslant s^2 x^* x s^2 = s^2 x^*(s^2 x^*)^*$ follows, which gives $s^2 x^* \in V$ and, hence, $(s^2x^*)^* s^2 x^* = xs^4x^* \in V$, as wished. A straightforward linearisation on s and $1 - s \in V$ gives $xsx^* \in V$. Now let $x, y \in V$. We have $s = yy^* \in V$, so that $xsx^* = xyy^*x^* = xy(xy)^* \in V$, and consequently $xy \in V$. We proceed to show that for each $0 \neq a \in R$ either a or a^{-1} belongs to V. For if $a \notin V$ then $aa^* \gg 1$. It follows that $(a^{-1})^* a^{-1} \cdot aa^* \gg (a^{-1})^* a^{-1}$, placing $(a^{-1})^*$ in V and, hence, $a^{-1} \in V$. We are left with the assertion about the center Z_b of the ring V. We use here the observation due to S. Berberian that for each $x \in R$, the element $(1/(1 + xx^*))x$ is bounded relative to the natural ordering and, hence, this element is bounded according to our ordering. If $a \in Z_b$, then a commutes with $(1/(1 + xx^*))x$ and a commutes with the bounded element $(1/(1 + xx^*)$. Thus a commutes with x, placing a in Z. It is now evident that Z_b is a valuation $*$-subring of the field Z.

(3) If x is a unit (e.g., $r_0 \leqslant xx^* \leqslant r_1$ for some rationals r_i) it is clear that $x \in V$. Also $xx^* \geqslant r_0$ gives $xx^*(x^*)^{-1}x^{-1} \geqslant r_0(x^*)^{-1}x^{-1}$ so that $(x^*)^{-1}x^{-1} \geqslant r_0^{-1}$. From this, $(x^{-1})^* \in V$ and hence $x^{-1} \in V$. Thus x is a unit of the ring V. Conversely, if x is a unit of the ring V the process can be reversed to get that x is a unit. It is now clear that the unique maximal right ideal of the local ring V coincides with the infinitesimals J or R.

(4) All we have to show is that if $s = s^*$ and $k = -k^*$ belong to V then $sk - ks \in J$. By Remark 5, we may assume that $s^2 \; O \; -k^2$ and, consequently, s, k are units of V. It follows that $|s| \; O \; (-k^2)$. By a routine argument, we can find a rational q_0 with $0 < |s| + q_0 k^2 < -k^2$. If s is replaced by $|s| + q_0 k^2$ this does not change the commutator nor does it affect the assumption on s

and k. If, further, we suppose that $-k^2 \leqslant \frac{1}{2}$ then, making use of Remark 4, we have

$$\begin{aligned}|sk-ks| &= |(|s|+q_0k^2)k - k(|s|+q_0k^2)| \\ &\leqslant (|s|+l_0k^2)^2 - k^2 \leqslant (-k^2+\varepsilon) - k^2 \leqslant -3k^2,\end{aligned}$$

where ε is any rational smaller than $-k^2$. Replacing k by rk with r any rational between 0 and 1 and eliminating shows that $|sk-ks| \ll -k^2$, so $sk-ks \in J$. For general k, it suffices to pass to $k/2r$, where r is any rational such that $-k^2 \leqslant r^2$.

In [6], Holland studied those subrings A of the division ring R which, in addition to being *-closed valuation subrings, contain in their group of units all elements of the form x^*x^{-1}, $0 \neq x \in R$. We call these subrings, with Holland, **-valuation subrings*. Unlike the situation for general Baer ordering we shall be able to show that the considered bounded subring V is a *-valuation subring. This will require the deeper study of the units of V, which is carried out in the next section. In order to prepare this study we establish here a characterization of the residue congruence relation $a \equiv b \pmod J$ $(a, b \in V)$ in terms of the rational bounds of a, b. This is the

THEOREM 5. *Let a, b be bounded elements with b a positive symmetric. If a and b admit the same rational upper and lower bounds then $\frac{1}{2}(a+a^*) \equiv b \pmod J$.*

Proof. If b is an infinitesimal, then $0 < b < \varepsilon$ for each rational ε. From this, $0 \leqslant a \leqslant \varepsilon$, and consequently $0 \leqslant \frac{1}{2}(a+a^*) \leqslant \varepsilon$, giving $\frac{1}{2}(a+a^*)$ an infinitesimal. If, on the other hand, b is a unit, we claim that $\frac{1}{2}(a+a^*)$ and b have the same rational upper and lower bounds. For if, for instance, $\frac{1}{2}(a+a^*) \geqslant r$ but $b \ngeqslant r$, then $b < r$, and so $a \leqslant r$, whence $\frac{1}{2}(a+a^*) = r$. If now $b < r - \varepsilon$, then $\frac{1}{2}(a+a^*) \leqslant r - \varepsilon < r$, which is impossible. This shows that $r - b \geqslant 0$ has no rational lower bounds. Thus $r - b$ is an infinitesimal, which gives $b \equiv r \equiv \frac{1}{2}(a+a^*) \pmod J$. Thus we may assume that $\frac{1}{2}(a+a^*) \geqslant r$ implies $b \geqslant r$. The converse is obvious. Therefore b and $\frac{1}{2}(a+a^*)$ are positive symmetrics with identical rational bounds. If now these elements are not congruent modulo J, then $b - c \notin J$, where $c = \frac{1}{2}(a+a^*)$. Say $b - c > 0$. Thus $b - c \geqslant \varepsilon_0$ for some rational ε_0. Because $c \notin J$, $c \geqslant \varepsilon_1$ for some rational ε_1. Now

$$b = (b-c) + c \geqslant \varepsilon_0 + \varepsilon_1,$$

$$c = \varepsilon_0 + \varepsilon_1,$$

$$b \geqslant \varepsilon_0 + (\varepsilon_0 + \varepsilon_1) = 2\varepsilon_0 + \varepsilon_1,$$

$$c \geqslant 2\varepsilon_0 + \varepsilon_1,$$

$$b \geqslant \varepsilon_0 + (2\varepsilon_0 + \varepsilon_1) = 3\varepsilon_0 + \varepsilon_1, \ldots$$

$$b \geqslant k\varepsilon_0 + \varepsilon_1,$$

where k is an arbitrary natural number, which contradicts the boundedness of b. The theorem is proved.

For the rest of this section, we shall examine one possible c-ordering on the division ring $\bar{V} = V/J$, which is basically induced by the ground c-ordering on R. Clearly, if we are to make $a + J$ positive in $\bar{V} = V/J$ relative to some c-ordering, then $a + J \neq 0$. Also, since the only skew element of $\bar{V}$, which is comparable to 0, is 0 itself we must have $a + a^* + J \neq 0$, which implies evidently $a + J \neq 0$. Finally, it is natural that those elements $a = a^*$ in V with $a \geqslant 0$ have an image $a + J \geqslant 0$. This leads to the notion of positive element modulo J as indicated in the following theorem.

THEOREM 6. *Call x a positive element modulo J if and only if $x = a + j$, where a is a positive bounded elements, j is an infinitesimal, and $a + a^*$, is a unit. In V/J define $x + J$ to be positive if and only if x is positive modulo J. This determines a c-ordering on the division ring V/J such that $b = b^* \geqslant 0$ with $b \in V$ implies $b + J \geqslant 0$. Under this c-ordering, V/J has all its elements $\neq 0$ units (archimedean c-ordered division ring).*

Proof. If a, b are positive elements with $a + a^*$, $b + b^*$ units then so must $a + b$. For $(a + a^*) + (b + b^*) = (a + b) + (a + b)^* \geqslant r_0 + r_1$, where r_0, r_1 are rational lower bound of $a + a^*$, $b + b^*$. This shows that $\mathscr{M} + \mathscr{M} \subseteq \mathscr{M}$, where $\mathscr{M}$ denotes all positive elements modulo J. Clearly $\mathscr{M} = \mathscr{M}^*$ and $\mathscr{M} \cap J = 0$, $1 \in \mathscr{M}$. If $y \notin J$ and a is a positive bounded elements as before, then $ayy^* > 0$. Also $ayy^* + (ayy^*)^* \equiv 2ayy^* \notin J$ (for by Theorem 4, all norms in V are central modulo J). Thus ayy^* is a positive element modulo J. From this $\mathscr{M}yy^* \subseteq \mathscr{M}$, for each $y \notin J$. If $\bar{\mathscr{M}} = \mathscr{M}/J$, it is clear that $\bar{\mathscr{M}}$ induces a c-ordering on V/J. For $b = b^* \in V$, clearly $b + J \geqslant 0$ whenever $b \geqslant 0$. By construction, for each $x \in V - J$, $xx^* \geqslant r_0$, for some rational r_0. Thus $\bar{x}\bar{x}^* \geqslant r_0$. This shows that each $\bar{x} \in V/J$, $\bar{x} \neq 0$, is a unit.

Let $(V/J)^{\text{sym.}}$ be the subset of symmetric elements of the division ring V/J. This is a subfield which is q-ordered by the restriction of the induced ordering. By a result of Prestel, the archimedean q-ordered field $(V/J)^{\text{sym.}}$ is an ordered field. Equivalently, if $a = a^*$ and $b = b^*$ are positive elements modulo J then ab is positive modulo J. This is the

COROLLARY. *If a and b are symmetric positive elements modulo J then ab is positive modulo J.*

3. More on the Bounded Elements

In this section we shall establish that the ring V of bounded elements is a *-valuation subring of R (in the sense of Holland). This means that the valuation *-subring V contains all the elements x^*x^{-1} in its group of units. Equivalently, V is preserved under conjugation. In fact, we prove a much stronger result, namely, for each $0 \neq x \in R$ and each bounded element $a = a^*$ of V, we have $xax^{-1} \equiv a \pmod J$. This gives the equivalence law $yy^* \approx y^*y$ for all $y \in R$, which expresses continuity of the involution (see [6]). In order to facilitate the writing we make the

Definition 5. If for each $a = a^* \in V$ and for a fixed $0 \neq x$, we have

$$xax^{-1} \equiv a \pmod J \qquad (\text{resp. } xax^{-1} \equiv a^* \pmod J),$$

we shall say that x *induces the identity* (resp. *the involution*) *modulo J.*

Lemma 1. *In order for $x \in P$ to induce the identity or the involution modulo J it is necessary and sufficient that*

$$\tfrac{1}{2}(xsx^{-1} + (x^{-1}sx)^*) \equiv s \qquad (\operatorname{mod} J),$$

for each positive symmetric unit s of V provided the residue division ring $\bar{V} = V/J$ is a field.

Proof. The condition is evidently necessary. Conversely, if this condition is satisfied then for $s = s^*$ a positive infinitesimal, $1 + s$ is a positive unit. Thus $\frac{1}{2}(x(1+s)x^{-1}) + (x(1+s)x^{-1})^*] = 1 + (\frac{1}{2}(xsx^{-1} + (xsx^{-1})^*)) \equiv 1 + x \pmod J$, and, hence, $\frac{1}{2}(xsx^{-1} + (xsx^{-1})^*) \equiv s \bmod J)$. Since every symmetric is positive or negative, we see that the congruence $\frac{1}{2}(xsx^{-1} + (xsx^{-1})^*) \equiv s \pmod J$ holds identically for each $s = s^* \in V$. We show next that $xsx^{-1} \in V$ for all $s = s^* \in V$. It suffices to show this for s any positive unit in V. In this case, if $xsx^{-1} \notin V$, necessarily $(xsx^{-1})^{-1} = xs^{-1}x^{-1} \in J$. Because s is a positive unit we have $s^{-1} \geqslant r_0$ for some rational r_0. Then $s^{-1} - r_0 \geqslant 0$ and, hence, $x(s^{-1} - r_0)x^{-1} \geqslant 0$ giving $xs^{-1}x^{-1} \geqslant r_0$. From this, $t = t^* = xs^{-1}x^{-1} + (xs^{-1}x^{-1})^* \geqslant 2r_0$, which contradicts the relation $xs^{-1}x^{-1} \in J$. This shows that $xsx^{-1} \in V$. We can now show that $xVx^{-1} \subseteq V$. For if $k = -k^* \in V$ then $xk^2x^{-1} = (xkx^{-1})^2 \in V$. By Corollary 1 to Theorem 4, $xkx^{-1} \in V$ follows.

Consider the mapping $\phi\colon a + J \to xax^{-1} + J$. This is a well defined automorphism of the division ring $\bar{V} = V/J$. Also $\frac{1}{2}(\phi + \phi^*)$ fixes all symmetrics of $\bar{V}$. Using the fact that the symmetrics of $\bar{V}$ form a subfield of $\bar{V}$ (Theorem 4) and by a routine linearization argument, one can show that ϕ fixes all symmetrics of $\bar{V}$. If $\bar{V}$ is a field coinciding with its symmetric part, $\phi = \mathrm{id}_{\bar{V}}$ follows. If, on the other hand, $\bar{V}$ is a field not coinciding with its symmetric part, it has exactly two automorphisms fixing the symmetrics so that $\phi = \mathrm{id}_{\bar{V}}$ or ϕ is the involution on $\bar{V}$ which completes the proof.

Given $x \in P$ and $s = s^* \in V$, it was shown in the course of the proof of Lemma 1 that if $s \geqslant q$, q rational, then $xsx^{-1} \geqslant q$ and conversely. By Theorem 5, $s \equiv \frac{1}{2}(xsx^{-1} + (xsx^{-1})^*)$ follows. In view of Lemma 1, x induces the identity or the involution modulo J. This is

LEMMA 2. *For $x \in P$, x induces the identity or the involution modulo J.*

In order to carry over Lemma 2 from the core elements $x \in P$ to arbitrary symmetrics $x = x^*$ much more work is needed. For one thing, we have no control on the rational majorants of $\frac{1}{2}(xsx^{-1} + (xsx^{-1})^*)$ as compared to those of s. For another, the conclusion of Lemma 2 is highly non-linear in nature. Hence the arduous lemmas that follow.

LEMMA 3. *Let $s = s^*$ and $k = -k^*$ be such that*

$$|sk - ks| \not\leqslant s^2 - k^2.$$

We have

(1) *sk^{-1} is a unit of the ring V,*

(2) *$sk^{-1} - k^{-1}s$ is a unit of the ring V,*

(3) *$sk - ks$ and $k(sk - ks)\,k^* = k(ks - sk)k$*

have opposite signs.

Proof. (1) If $sk^{-1} = j \in J$ then $s^2 = (jk)^2 = (jk)(jk)^* = -jk^2j^*$. For each rational r, if $s^2 \geqslant r_0(-k^2)$ then $-jk^2j^* = s^2 \geqslant r_0(-k^2)$ gives $jk^2j^*k^{-2} = -jk^2j^*(-k^{-2}) \geqslant r_0(-k^2)(-k^2) = r_0$. From this,

$$t = jk^2j^*k^{-2} + (jk^2j^*k^{-2})^* \geqslant 2r_0.$$

But, from $j^* \in J$ and $-k^2 \in P$ follow $k^2j^*k^{-2} \in J$ (Theorem 7) and, consequently, $t = t^* \in J$, contradicting the inequality $t \geqslant 2r_0$. This shows that $s^2 \leqslant -k^2$. In view of Remark 4, $|sk - ks| \leqslant s^2 - k^2$ follows, which contradicts our assumption. We must have $sk^{-1} \notin J$. Similarly $ks^{-1} \notin J$. Therefore sk^{-1} is a unit of V.

(2) By (1), $sk^{-1} - k^{-1}s = sk^{-1} + (sk^{-1})^* \in V$. Suppose, by way of contradiction, that $j = sk^{-1} - k^{-1}s \in J$. We have

$$\begin{aligned}(sk - ks) + (k^2sk^{-1} - k^{-1}sk^2) &= (sk^{-1} - k^{-1}s)\,k^2 + k^2(sk^{-1} - k^{-1}s)\\ &= jk^2 + k^2j;\\ sk - ks &= jk^2 + k^2j - (k^2sk^{-1} - k^{-1}sk^2).\end{aligned}$$

Since $j = j^* \in J$ it follows that $-|j|\, k^2 \leqslant -k^2$ and $-k^2\,|j| \leqslant -k^2$. Suppose that

$$h = |\, k^2 s k^{-1} - k^{-1} s k^2 \,| > r_0(-k^2),$$

for some positive rational $r_0 > 0$. If $-k^2$ induces the identity mapping modulo J then

$$h(-k^2) = |k^2 s k^{-1} k^{-2} - k^{-1} s| > r_0,$$

with $k^2 s k^{-1} k^{-2} \equiv s k^{-1} \pmod{J}$. It would follow that

$$r_0 < |\, k^2(sk^{-1})\, k^{-2} - k^{-1} s| \equiv |sk^{-1} - k^{-1} s| = |j| \in J,$$

which is a contradiction. If, on the other hand, $-k^2$ induces the involution modulo J, then

$$\begin{aligned} r_0 < h(-k^{-2}) &= |\, k^2(sk^{-1})\, k^{-2} - k^{-1} s| \\ &\equiv |-2k^{-1} s|. \end{aligned}$$

It would follow that

$$\begin{aligned} 2r_0 < h(-k^{-2}) + (-k^2)k &\equiv 2\,|k^{-1} s - s k^{-1}| \\ &= 2j, \end{aligned}$$

which is also a contradiction. We have shown that $h \leqslant -k^2$. Thus $|sk - ks| \leqslant |\, jk^2 + k^2 j| + h \leqslant -k^2$, which contradicts the hypothesis. We must have $sk^{-1} - k^{-1} s \in V - J$.

(3) Suppose, by way of contradiction, that $sk - ks$ and $k(sk - ks)k$ have opposite signs. Say $sk - ks > 0$ and $k(sk - ks)k < 0$. We claim that $-k^2$ cannot induce the involution modulo J. For otherwise set $t = sk^{-1} - k^{-1} s$. From $k(sk - ks)k < 0$ follows $(-k^{-2})(k(sk - ks)k)(-k^{-2}) = k^{-1}(sk - ks)\, k^{-1} = k^{-1} s - sk^{-1} = -t < 0$, that is, $t > 0$. But then $k^{-1} t k^{-1} = k^{-2}(ks - sk)\, k^{-2} = (-k^{-2})(ks - sk)(-k^{-2}) < 0$ and consequently $k^{-1} t k = -k^{-1} t k^{-1}(-k^{-2}) > 0$. Now

$$\begin{aligned} 0 < k^{-1} t k &= k^{-1}(sk^{-1} - k^{-1} s)k \\ &= k^{-1} s - k^{-2}(sk^{-1})\, k^2 \\ &\equiv k^{-1} s + k^{-1} s = 2(k^{-1} s). \end{aligned}$$

Thus $2(k^{-1} s - sk^{-1}) \equiv k^{-1} t k + (k^{-1} t k)^* > 0$, that is, $-2t \equiv k^{-1} t k + (k^{-1} t k)^* > 0$, which together with the relation $t > 0$ gives $t \in J$. However, from the above, $sk^{-1} - k^{-1} s \notin J$. This shows that $-k^2$ must induce the identity modulo J. Since $k^{-1}s + sk^{-1} \in V$ it follows that $k^{-2}(k^{-1}s + sk^{-1})\, k^2 = (k^{-1}s + sk^{-1}) + j$, for some $j \in J$. Thus

$$(k^{-1}s + sk^{-1})\, k^2 - k^2(k^{-1}s + sk^{-1})$$
$$= (sk - ks) + (k^{-1}sk^2 - k^2sk^{-1}) = jk^2.$$

Now $k^{-1}sk^2 - k^2sk^{-1} = (k^{-1}s - sk^{-1})k^2 + (sk^{-1})k^2 - k^2(sk^{-1} = (k^{-1}s - sk^{-1})k^2 + j'k^2$, where $j' \in J$ (for $sk^{-1} \in V$). Then

$$\begin{aligned} sk - ks &= jk^2 - (k^{-1}sk^2 - k^2sk^{-1}) \\ &= jk^2 - ((k^{-1}s - sk^{-1})\, k^2 + j'k^2) \\ &= (j - j')\, k^2 + (k^{-1}s - sk^{-1})(-k^2). \end{aligned}$$

Since $k^{-1}s - sk^{-1} < 0$ it follows that $(k^{-1}s - sk^{-1})(-k^2) < 0$ and, consequently,

$$sk - ks \leqslant (j - j')\, k^2 = j''(-k^2), \qquad j'' \in J.$$

Then $2(sk - ks) \leqslant j''(-k^2) + (-k^2)(j'')^*$. Thus $v = v^* = j^*(-k^2) + (-k^2)(j)^* > 0$. Since $k^2j^*k^{-2} \in J$ it follows that $v \ll -k^2$. Consequently, $0 \leqslant sk - ks \ll -k^2$, which contradicts our hypothesis. With this the lemma is proved.

LEMMA 4. *Let $s = s^*$ be a unit of the ring V such that s and $1 - s$ belong to P, and let $0 \neq d = d^* \in R$. We have*

(1) $dsd^{-1} \in P \cap V$,

(2) $ds^2d \geqslant sd^2s$ *implies* dsd^{-1} *is a unit of* V, *and*

(3) $ds^2d \geqslant (1 + \varepsilon)\, sd^2s$ *for some rational* ε *implies* $2dsd > sd^2 + d^2s$.

Proof. (1) Let $\geqslant_1^{\text{nat.}}$ be the ordering of natural type (Definition 3). From the hypothesis, $0 <_1^{\text{nat.}} s <_1^{\text{nat.}} 1$ follows $(d^{-1}sd)(d^{-1}sd)^* <_1^{\text{nat.}} 1$ and consequently $d^{-1}sd$ is a bounded element, which is in P (Theorem 1, Section 1).

(2) Right multiplying the inequality by d^{-2} gives $ds^2d^{-1} \geqslant sd^2sd^{-2}$. By the theorem, we just proved $d^2sd^{-2} \equiv s \pmod J$. Therefore $ds^2d^{-1} \geqslant sd^2sd^{-2} \equiv s^2 \pmod J$. Since s^2 is a unit of the ring V, we get that $ds^2d^{-1} + (ds^2d^{-1})^* \geqslant 2s^2$ is a unit of the ring V. Then ds^2d^{-1} is a unit of V.

(3) If $t = t^* = 2dsd - (sd^2 + d^2s) \leqslant 0$ then right multiply this by d^{-2} to get

$$2dsd^{-1} < s + d^2sd^{-2} \equiv 2s.$$

Because dsd^{-1} belongs to P we get, using Theorem 3, Section 1, that

$$4ds^2d^{-1} \leqslant (s + d^2sd^{-2})^2 \equiv 4s^2.$$

From this

$$ds^2d^{-1} + d^{-1}s^2s < 2s^2 + \varepsilon',$$

for each rational ε'. By the assumption, we have $ds^2d^{-1} \geqslant (1+\varepsilon)\, sd^2sd^{-2}$, which gives

$$\begin{aligned} ds^2d^{-1} + d^{-1}s^2d &\geqslant (1+\varepsilon)(sd^2sd^{-2} + d^{-2}sd^2s) \\ &\equiv 2(1+\varepsilon)\, s^2 - 2s^2 + 2\varepsilon s^2. \end{aligned}$$

Since s^2 is a unit of V we get

$$ds^2d^{-1} + d^{-1}s^2d \geqslant 2s^2 + \varepsilon'',$$

for some rational ε'', which is a contradiction. This shows that $t = t^* > 0$.

LEMMA 5. *Under the assumption on s in Lemma* 4, *we have* $d^2sd^{-1} < (1+\varepsilon)s^2$ *for each* $0 \neq d = d^* \in R$.

Proof. Set

$$t = t(s, d) = d^{-1}sd + dsd^{-1} - 2s.$$

We show first that $t \in J$ provided $t \geqslant 0$ and $ds^2d \geqslant (1+\varepsilon)\, sd^2s$ for some rational $\varepsilon > 0$. By Lemma 4, $t \in V$. If then $t \notin J$, clearly $d^{-1}sd \not\equiv s \pmod J$.

Define $a = a(s, d) = sd + ds$ and $k = k(s, d) = sd - ds$. Define $s_r = s + r$ with $r \geqslant 1$ any natural number, and let $a_r = a(s_r, d)$ and $k_r = k(s_r, d) = k(s, d)$. We have

$$\begin{aligned} a_rk_r - k_ra_r = a_rk &= ka_r \\ &= (ak - ka) + r(2dsd - (sd^2 + d^2s)) \\ &= 2(ds_r^2d - s_rd^2s_r). \end{aligned}$$

By the hypothesis, $ds^2d \geqslant (1+\varepsilon)\, sd^2s$. In view of Lemma 4, part (3), $2dsd > sd^2 + d^2s$ follows. Since $ak - ka = 2(ds^2d - sd^2s) > 0$ and since $r(2dsd - (sd^2 - d^2s)) > 0$ it follows that $a_rk_r - k_ra_r > 0$ for all r. If now $k_r(a_rk_r - k_ra_r)\, k_r < 0$ for some r then, by Lemma 3, part (3), we would get

$$|a_rk_r - k_ra_r| \leqslant a_r^2 - k_r^2.$$

By Remark 6, Section 1, $xx^* \approx x^*x$ follows, where $x = a_r + k^r$. Thus $4ds_r^2d \approx 4s_rd^2s_r$. Then $ds_r^2\, d^{-1} \approx s_rd^2s_rd^{-2} \equiv s_r^2 \pmod J$. Because $s_r = s + r$ is a unit of V (for by Lemma 4, s is a unit of V) belonging to the *-core of R, we derive as before that $ds_rd^{-1} \equiv s_r \pmod J$ and consequently $dsd^{-1} \equiv s \pmod J$, which we agreed to rule out. This shows that $k(a_rk_r - k_ra_r)k$ and $a_rk_r - k_ra_r$ have the same sign, which is plus.

Define $\tau = k^{-1}(sd^2 + d^2s - 2dsd)\, k^{-1}$. From $k(a_rk - ka_r)k > 0$ follows

$$k^{-1}(a_rk - ka_r)\, k^{-1} = (-k^{-2})\, k(a_rk - ka_r)\, k(-k^{-2}) > 0.$$

Expanding this we get that

$$\begin{aligned} k^{-1}(ak - ka) + r(2dsd - (sd^2 + d^2s))\, k^{-1} \\ = k^{-1}a - ak^{-1} - rk^{-1}(sd^2 + d^2s - 2dsd)\, k^{-1} > 0, \\ k^{-1}a - ak^{-1} > rk^{-1}(sd^2 + d^2s - 2dsd)\, k^{-1} = r\tau, \end{aligned}$$

for each $r = 1, 2, \ldots$.

Now $k = sd - ds = ds(g-1)$, where $g = s^{-1}(d^{-1}sd)$. Because s is a unit and $d^{-1}sd$ is a unit so must be g. Thus $g - 1 \in V$. However,

$$g^* = dsd^{-1}s^{-1}$$

and $dsd^{-1} = d^2(d^{-1}sd)\, d^{-2}$. Since $d^{-1}sd \in V$ and since $d^2 = dd^*$ we get, using Lemma 2, that $dsd^{-1} \equiv d^{-1}sd \pmod J$. Because s^{-1} is central modulo J we see that $g^* \equiv d^{-1}sd \cdot s^{-1} \equiv s^{-1}d^{-1}sd \equiv g \pmod J$. Now if $g - 1 \in J$ we get $dsd^{-1}s^{-1} \equiv 1 \pmod J$ and hence $dsd^{-1} \equiv s \pmod J$, which we agreed to rule out. This shows that $g - 1$ is a unit, which is symmetric modulo J. Thus

$$\begin{aligned} \tau &= k^{-1}(sd^2 + d^2s - 2dsd)\, k^{-1} \\ &= (g-1)^{-1}s^{-1}d^{-1}(sd^2 + d^2s - 2dsd)\, d^{-1}s^{-1}(g-1)^{-1} \\ &= (g-1)^{-1}s^{-1}(d^{-1}sd + dsd^{-1} - 2s)\, s^{-1}(g-1)^{-1} \\ &= (g-1)^{-1}s^{-1}ts^{-1}(g-1)^{-1} \end{aligned}$$

is a unit of V, which is a symmetric modulo J. Because $s^{-1}ts^{-1}$ is a positive symmetric unit of V we get that $s^{-1}ts^{-1}$ maps onto a positive symmetric of $\bar{V} = V/J$. Thus τ maps onto the positive symmetric

$$\bar{\tau} = \overline{(g-1)^{-1}(s^{-1}ts^{-1})((g-1)^{-1})^*}.$$

Because $\tau = \tau^*$ is a unit of V it follows that τ is positive and, consequently. $\tau \geqslant r_0'$, for some rational r_0. Recalling that $k^{-1}a - ak^{-1} > r\tau$ for each natural number $r = 1, 2, \ldots$ we get that $k^{-1}a - ak^{-1}$ is infinitely large. By the contraposition of Lemma 3, part (2), this gives that $|ak - ka| \ll a^2 - k^2$. Setting $x = a + k$ and using Remark 6, Section 1, we then get

$$\begin{aligned} 4ds^2d = x^*x &\approx xx^* = 4sd^2s \\ ds^2d^{-1} &\approx s^2, \\ dsd^{-1} &\equiv s \qquad (\mathrm{mod}\, J), \end{aligned}$$

which we agreed to rule out. This shows that t was in J, as wished.

More generally, if $ds^2d \geqslant (1+\varepsilon)\, sd^2s$ for some rational ε, we claim that s^2 satisfies the conditions in the above:

$$t(s^2, d) \geqslant 0, \qquad ds^4d \geqslant (1+\varepsilon)\, s^2d^2s^2.$$

For then $ds^2d^{-1} \geqslant (1+\varepsilon)\, sd^2sd^{-2}$, giving $ds^4d^{-1} \geqslant (1+\varepsilon)^2(sd^2sd^{-2})^2$ (Theorem 3), and $(sd^2sd^{-2})^2 \equiv (1+\varepsilon)^2 s^4$. This rules out the inequality $ds^4d < (1+\varepsilon_0)\, s^2d^2s^2$ for every rational ε_0. Thus $ds^4d > (1+\varepsilon')\, s^2d^2s^2$ for some rational ε'. Also if, by way of contradiction,

$$t(s^2, d) = d^{-1}s^2d + ds^2d^{-1} - 2s^2 < 0,$$

then from $ds^2d > (1+\varepsilon)\, sd^2s$ we would get

$$\begin{aligned} ds^2d^{-1} &> (1+\varepsilon)\, sd^2sd^{-2}, \\ d^{-1}s^2s &> (1+\varepsilon)\, d^{-2}sd^2s \\ ds^2d^{-1} + d^{-1}s^2d &> (1+\varepsilon)(sd^{-2} + d^{-2}sd^2s); \end{aligned}$$

passing to $\bar{V} = V/J$ this gives

$$\overline{ds^2d^{-1} + d^{-1}s^2d} \geqslant (1+\varepsilon)(2\bar{s}^2).$$

On the other hand, $t(s^2, d) < 0$ gives

$$\overline{ds^2d^{-1} + d^{-1}s^2s} \leqslant \overline{2s^2},$$

which is a contradiction. This shows that $d(s^2)^2d \geqslant (1+\varepsilon)\, s^2d^2s^2$, for some rational ε, implies $t(s^2, d) > 0$. By the above $t(s^2, d) \in J$, that is, $d^{-1}s^2d + ds^2d^{-1} \equiv 2s^2 \pmod{J}$. Recalling that $d^{-1}s^2d \equiv ds^2d^{-1} \pmod{J}$ we would get $d^{-1}s^2d \equiv s^2 \pmod{J}$, which forces $d^{-1}sd \equiv s \pmod{J}$. This relation can be played off with the assumption $ds^2d \geqslant (1+\varepsilon)\, sd^2s$. The upshot of this is that $ds^2d < (1+\varepsilon)\, sd^2s$ for every rational ε. Equivalently

$$\begin{aligned} ds^2d^{-1} &< (1+\varepsilon)\, sd^2sd^{-2} \equiv (1+\varepsilon)\, s^2, \\ ds^2d^{-1} &< (1+\varepsilon)\, s^2, \end{aligned}$$

for each rational ε, which completes the proof.

LEMMA 6. *For each* $0 \neq d = d^*$ *and each* $s = s^* \in V$ *we have* $dsd^{-1} \equiv s \pmod{J}$. *Therefore* d *induces the identity on the involution modulo* J.

Proof. Let $s = s^*$ be as in Lemma 5. We have $dsd^{-1} \in V$ and

$$d^{-1}sd = d^{-2}(dsd^{-1})\,d^2 \equiv dsd^{-1} \qquad (\operatorname{mod} J).$$

Now $ds^2d^{-1} < (1+\varepsilon)^2 s^2$ gives

$$\tfrac{1}{2}(d^{-1}s^2d + ds^2d^{-1}) \equiv ds^2d^{-1} < (1+\varepsilon)^2 s^2;$$

then

$$\begin{aligned}\tfrac{1}{2}(d^{-1}sd + dsd^{-1})^2 &\equiv \tfrac{1}{4}(2(dsd^{-1}))^2\\ &= ds^2d^{-1} < (1+\varepsilon)^2 s^2.\end{aligned}$$

Going down to V/J and using the fact that the symmetrics form an ordered subfield, we get

$$\tfrac{1}{2}(d^{-1}sd + dsd^{-1}) < (1+\varepsilon)^2 s.$$

Replacing s by $\frac{1}{2}(dsd^{-1} + d^{-1}sd)$, we get

$$\begin{gathered}\tfrac{1}{4}[(d^2sd^{-2} + s) + s + d^2sd^{-2}] < (1+\varepsilon)(dsd^{-1} + d^{+1}sd)\tfrac{1}{2},\\ s \equiv \tfrac{1}{4}[(d^2sd^{-2} + s) - (s + d^{-2}sd^2)] < ((1+\varepsilon)/2)(dsd^{-1} + d^{-1}sd),\\ d^{-1}sd \equiv \tfrac{1}{2}(dsd^{-1} + d^{-1}sd) \equiv s \qquad (\operatorname{mod} J).\end{gathered}$$

Given $d = d^* \in R$ and $s = s^* \in V$, we claim that $ds^2d^{-1} \equiv s^2 \pmod{J}$. For $s^2 \in V \cap P$ and $s_1 = 1/(1+s^2) \in P \cap V$ with $s_1 \leqslant_1^{\text{nat.}} 1$. By Lemma 5, $ds_1d^{-1} \equiv s_1 \pmod{J}$ follows. Notice that s_1 is an invertible element of V. Since $ds_1d^{-1} \equiv s_1 \pmod{J}$ it follows that $ds_1d^{-1} \in V - J$, so that ds_1d^{-1} is invertible in V. Then $(ds_1d^{-1})^{-1} \equiv s_1^{-1} \pmod{J}$, that is,

$$d(1+s^2)\,d^{-1} \equiv 1 + s^2 \qquad (\operatorname{mod} J);$$

equivalently $ds^2d^{-1} \equiv s^2 \pmod{J}$. The usual linearization argument shows that $dsd^{-1} \equiv s \pmod{J}$. As in the proof of Lemma 1, we conclude that d induces the identity mapping or the involution modulo J.

LEMMA 7. *For each* $x \in R$, *we have* $xx^* \approx x^*x$.

Proof. By Remark 6, it suffices to prove that $|sk - ks| \ll s^2 - k^2$, for each $s = s^*$ and $k = -k^*$. Assume this not to be true for the pair s, k. By

Lemma 3, parts (1), (2), we would get $s^{-1}k$, ks^{-1} and $s^{-1}k-ks^{-1}\in V-J$. For $x=s=s^*$ we have

$$s(s^{-1}k-ks^{-1})\,s^{-1}\equiv s^{-1}k-ks^{-1} \qquad (\operatorname{mod} J).$$

But,

$$\begin{aligned} s(s^{-1}k-ks^{-1})\,s^{-1} &= ks^{-1}-s(ks^{-1})\,s^{-1} \\ &= \begin{cases} ks^{-1}-ks^{-1}=0 \\ \qquad\text{or} \\ ks^{-1}+s^{-1}k, \end{cases} \end{aligned}$$

accordingly as s induces the identity or the involution modulo J. In both cases, we get $s^{-1}k-ks^{-1}\equiv 0$ (mod J), which contradicts the invertibility of $s^{-1}k-ks^{-1}$ modulo J.

COROLLARY 1. *Each $x=-x^*\neq 0$ induces the identity or the involution modulo J.*

Proof. Take $y=sk$ with $s=s^*$, $k=-k^*$. Then $-sk^2s=yy''\approx y^*y=-ks^2k$. From this $-sk^2s(-k^{-2})=sk^2sk^{-2}\approx -ks^2k(-k^{-2})=ks^2k^{-1}$. Choosing $s\in V$ we have, modulo V,

$$s^2\equiv sk^2sk^{-2}\approx ks^2k^{-1},$$

and consequently $ks^2k^{-1}\equiv s^2$ (mod V). As before, we conclude that k satisfies the desired conclusion.

LEMMA 8. *Each pseudo unitary element $x^{-1}x^*$ belongs to V. In fact, $x^{-1}x^*$ is a unit of V.*

Proof. We have, for $x+x^*\neq 0$,

$$\begin{aligned} &(x+x^*)\,x^{-1}x^*(x+x^*)^{-1} \\ &\quad = x(1+x^{-1}x^*)\,x^{-1}x^*(1+x^{-1}x^*)^{-1}x^{-1} \\ &\quad = xx^{-1}x^*x^{-1}=x^*x^{-1}. \end{aligned}$$

If then $x^{-1}x^*\in J$, clearly $x+x^*\neq 0$, so that $x^*x^{-1}\in(x+x^*)J(x+x^*)^{-1}\subseteq J$ (Lemma 6). But, $(x^{-1}x^*)^{-1}=(x^*)^{-1}x=(x^*x^{-1})^*$. Thus $((x^{-1}x^*)^{-1})^*=x^*x^{-1}\in J$, and consequently $(x^{-1}x^*)^{-1}\in J$, which contradicts the relation $x^{-1}x^*\in J$.

We must conclude that $x^{-1}x^*\notin J$. Similarly $(x^{-1}x^*)^{-1}=(x^*)^{-1}x=(x^*)^{-1}(x^*)^*\notin J$. Therefore $x^{-1}x^*$ is a unit of V.

We can now prove an important theorem.

THEOREM 7. *Let R be any c-ordered division ring. For each $0 \neq x \in R$ and each symmetric $s \in V$ we have $xsx^{-1} \equiv s \pmod{J}$. Therefore x induces the identity mapping or the involution modulo J provided the residue division ring V/J is a field. In all cases, V and J are preserved under conjugation.*

Proof. By Lemma 8, $x^*x^{-1} \in V$. Thus one of the elements $1 + x^{-1}x^*$, $1 - x^{-1}x^*$ is a unit of V. Now any unit u of V is such that $usu^{-1} \equiv s \pmod{J}$. For, since all symmetrics in V are central in V/J, we get

$$usu^{-1} - s = (us - su)\, u^{-1} \in J \cdot u^{-1} \subseteq J.$$

From Lemmas 5 and 6, the elements $x(1 + x^{-1}x^*) = x + x^*$, $x(1 - x^{-1}x^*) = x - x^*$ leave fixed symmetrics modulo J. Since one of the right factors $1 + x^{-1}x^*$, $1 - x^{-1}x^*$ leaves fixed the symmetrics modulo J, so must $x(1 + x^{-1}x^*)(1 + x^{-1}x^*)^{-1} = x$ or $x(1 - x^{-1}x^*)(1 - x^{-1}x^*)^{-1} = x$, which completes the proof.

COROLLARY 1. *If $s = s^*$ is any positive unit in V then $xsx^* \geqslant 0$ for each $x \in R$.*

Proof. We have $xsx^{-1} = s + j$, some $j \in J$. If now $xsx^* < 0$ then $xsx^{-1} < 0$. Thus $s = xsx^{-1} - j < -j$. From this, $0 < s < -(j + j)^* \in J$ giving $s \in J$, which contradicts the hypothesis.

COROLLARY 2. *V is a *-valuation subring of R.*

4

4a. *Algebraicity*

As is well known and easily determined, every ordered group is torsion-free. The analog for division rings of this result asks whether every Hilbert ordered division ring is a purely transcendental extension of its center Z. The answer to this question is "yes."

Let a be an algebraic element of the division ring R. If $p = p_a(t)$ is the minimal polynomial of a then, by a result of Wedderburn,

$$p = (t - a)(t - xax^{-1}) \cdots (t - vav^{-1}),$$

where $x,\dots, v \in R$. Then

$$\operatorname{trace}(a) = a + xax^{-1} + \cdots + vav^{-1}$$

and a has the same sign relative to any fixed Hilbert ordering or R. However, if a is algebraic so must be $a - (1/n)\operatorname{trace}(a)$, where n is the degree of p, and $\operatorname{trace}(a-(1/n)\operatorname{trace}(a))=0$. Therefore $a=(1/n)\operatorname{trace}(a)\in Z$.

Now let R be a division ring with involution and suppose that $a \geqslant b$ is both a Baer ordering and a c-ordering. If $s=s^*$ is an algebraic element of R then $(1/n)\operatorname{trace}(s)$ is a symmetric central element so that $s-(1/n)\operatorname{trace}(s)$ is a symmetric. If $s-(1/n)\operatorname{trace}(s)>0$ then $\operatorname{trace}(s-(1/n)\operatorname{trace}(s))>0$ follows, contradicting the relation

$$\operatorname{trace}(s-(1/n)\operatorname{trace}(s))=0.$$

Similarly $s-(1/n)\operatorname{trace}(s) \nless 0$. Therefore $s=(1/n)\operatorname{trace}(s)\in Z$. We have shown the

Remark 8 (Albert). (1) If R is a Hilbert ordered division ring then R is a purely transcendental extension of the center Z.

(2) If R has a joint Baer and c-ordering then every algebraic symmetric is central.

Since the nature of the center (rational function field or algebraic number field) turns out to be determinant to the existence or nonexistence of odd-dimensional division rings having a Baer ordering [6], it is legitimate to investigate the case of any c-ordered division ring whose center is an algebraic number field Z. Then the central bounded elements Z_b coincide with Z. More generally, we shall condition the algebraicity of individual elements a of R as follows. We suppose that a is algebraic over a subfield F of the valuation domain Z_b. For instance, if $F=F^*$ is totally archimedean in the sense given by D. Handelman [4],

$$\forall h\in F\,\exists h_i\in F\,\exists r\text{ (rational) } hh^*+\sum h_ih_i^*=r,$$

then $F\subseteq Z_b$ follows.

Remark 9. For F any subfield of Z_b, if x is algebraic over F then the minimal polynomial $p_x(t)$ of x over F and the minimal polynomial $g_x(t)$ of x over $Z\supseteq F$ have the properties

(1) $p_x, g_x\in Z_b[t]$, where $Z_b=Z\cap V$, is the center of V,

(2) g_x divides p_x over Z_b.

Justification. (1) Since x is algebraic over F it follows that x (resp. x^{-1}) is a polynomial expression of x^{-1}(resp. x) over $F\subseteq V$. From this and the relation x or $x^{-1}\in V$ (for V is a valuation domain) follow that both x and x^{-1} belong to V, so that x is a unit of V. (2) By Wedderburn's result, we have

$$g_x=\prod_1^n (t-a_ix_ix_i^{-1}) \qquad (a_i\in R).$$

Because V is preserved under conjugation, the relation above clearly shows that $g_x \in Z[t] \cap V[t] \subseteq Z_b[t]$. Since $p_x(x) = 0$ and since g_x is the minimal polynomial of x over $Z \supseteq F$, we have $p_x = hg_x$ for some $h \in Z[t]$. Write

$$p_x = \sum p_i t^i, \qquad h = \sum h_i t^i, \qquad g_x = \sum g_i t^i,$$

$(p_i, g_i \in Z_0)$. Notice that $p_0 \in F \subseteq Z_b$ is a unit of the valuation domain Z_b in Z. Now $g_0 = (-1^{\;n} \prod_{i=1}^{n} a_i x a_i^{-1}$. Because x is a unit of V so must g_0. Thus g_0 is a unit of Z_0. From the relation $p_0 = h_0 g_0$ it follows that h_0 is a unit of Z_b. From the relation $p_1 = h_1 g_0 + h_0 g_1$ with h_0, $g_1 \in Z_b$ and g_0 is a unit of Z_b follow $h_1 \in V$. Step by Step one can show that $h \in Z_b[t]$.

THEOREM 8. *Let R be any c-ordered division ring such that the residue division ring V/J be a field and let F be any *-closed central subfield of R, which is contained in V. (1) If $x \in R$ is algebraic over F then x is a bounded elements of degree at most 2 over the center Z. (2) Further, if $x = x^*$, or if some skew element $0 \neq \sigma \in Z$ is algebraic over F, then $x \in Z$.*

Proof. (1) Suppose, by way of contradiction, that the degree n of x over Z is larger than 2. By a result of Wedderburn, the minimal polynomial g_x of x over Z can be written in the form

$$g_x = (t - a_1 x a_1^{-1}) \cdots (t - a_n x a_n^{-1}).$$

For each $i = 1,..., n$, we have $a_i x a_i^{-1} \equiv x \pmod V$ or $\equiv x^* \pmod V$ for, by Remark 9, $x \in V$, and by Theorem 7, a_i induces the identity or the involution modulo J. Let $\bar{F} = F + J/J$, $\bar{x} = x + J$, and $\bar{x}^* = x^* + J$. Since the symmetrics of the division ring V/J are central it follows that $\bar{x}$ and $\bar{x}^*$ commute. Denote by $\bar{F}(\bar{x}, \bar{x}^*)$ the subfield of V/J generated by $\bar{F}$, $\bar{x}$, and $\bar{x}^*$. The polynomial g_x maps onto the polynomial $\bar{g}_x \in V/J[t]$ of degree n. for g_x is a monic polynomial, and we have over $\bar{F}(\bar{x}, \bar{x}^*)$ the factorisation

$$g_{\bar{x}} = (t - \bar{x}_1) \cdots (t - \bar{x}_n),$$

where $\bar{x}_i = \bar{x}$ or $\bar{x}^*$. Since $n > 2$, $\bar{g}_x$ has a multiple root in $\bar{F}(\bar{x}, \bar{x}^*)$.

If Z_0 is the center of the ring V, then Z_0 maps onto $\bar{Z}_0$. Denote by $\bar{Z}_0(\bar{x}, \bar{x}^*)$ the subfield of V/J generated by $\bar{Z}_0, \bar{x}, \bar{x}^*$. Clearly $\bar{Z}_0(\bar{x}, \bar{x}^*) \supseteq \bar{F}(\bar{x}, \bar{x}^*)$. Because g_x is a factor of the minimal polynomial p_x of x over F over Z_0, we get that $\bar{g}_x$ is a factor of $\bar{p}_x$ over $\bar{Z}_0(\bar{x}, \bar{x}^*)$. By the above, $\bar{p}_x$ has a multiple root in the extension field $\bar{Z}_0(\bar{x}, \bar{x}^*)$ of the field $\bar{F}$. Now $\bar{p}_x(\bar{x}) = 0$, and if $h(t) \in F[t]$ is such that $h(x) \in J$, we get that $h(x)$ is an algebraic element over F with $h(x) \in J$, resulting in $h(x) = 0$. This shows that $\bar{p}_x$ is the minimal polynomial of $\bar{x} = x + J$ over $\bar{F} \approx F$. But, in characteristic

0, it is well known that the minimal polynomial has no multiple roots over any field extension of $\bar{F}$, which contradicts the assumption $n > 2$.

(2) If $x = x^*$ we know from the proof of Theorem 7 that $a_i x a_i^{-1} \equiv x \pmod J$. The argument in the above can be re-played to get that the degree n of x over Z is equal to 1. Finally, if $0 \neq \sigma$ is a central skew element, which is algebraic over F, then σ is a central unit of V. From the congruence relations $aya^{-1} \equiv y \pmod J$, $y = y^* \in V$ follow quickly $a\sigma y a^{-1} \equiv \sigma y \pmod J$ giving $aha^{-1} \equiv h \pmod J$, for all $h \in V$ and $a \in R$. As before, we conclude that the degree of x over Z is equal to 1.

COROLLARY 1. *If the involution is of the second kind and if the *-center $(Z, *)$ of R is totally archimedean (for example, Z is algebraic over the rationals) then R is a purely transcendental extension of it center Z.*

Proof. Since $(Z, *)$ is totally archimedean it follows that $(Z, *) \subseteq V$. Because Z is at most quadratic over $(Z, *)$, $Z \subseteq V$ follows, that is, $Z = Z_b$. We can then quote Theorem 8.

COROLLARY 2. *Any symmetric of R which is algebraic over a totally Archimedean subfield $F = F^* \subseteq Z$ of R must be a central element.*

4.2. *Generation*

If U is the unitary group of the division ring R and $Z[U] = A$ is the span of the unitary group of R over the center Z, one would be tempted to think that such a subalgebra of R is a significative portion of R (for R with characteristic $\neq 2$). In fact, we asked (candidly) whether $Z[U] = R$ for R infinite dimensional over Z. Handelman has shown that if R has for center the reals $\mathbb{R}$ and R is not finite dimensional, then $A \neq R$. Still one might ask whether A is a simple ring. Also, what can be said about the over subalgebras $B \supseteq A$ containing the pseudo unitary elements of R(e.g., elements of the form $v = a^*a^{-1} \cdots t^*t^{-1}$, where $0 \neq a, \ldots, t \in R$)? Let us first give a considerably shorter proof of Handelman's result.

THEOREM 9 (Handelman). *Let R be any naturally orderable division ring (e.g., $\sum x_i x_i^* = 0$ implies $x_i = 0$) with center Z containing the reals $\mathbb{R} = \mathbb{R}^*$, let U be the unitary group of R, and let $A = \mathbb{R}[U]$ be the span of U over the field $\mathbb{R}$. Unless R is of dimension* 4 *(or less), we have $A \neq R$.*

Proof. Let $V^{\text{nat.}}$ (resp. $J^{\text{nat.}}$) be the subset of bounded elements (resp. infinitesimals) with respect to the natural ordering $a \geqslant^{\text{nat.}} b$ (see Definition 2). It is known that $V^{\text{nat.}}$ is a *-closed subring evidently containing U. Because $\mathbb{R}$ is totally archimedean subfield, $\mathbb{R} \subseteq V^{\text{nat.}}$ follows.

Therefore $V^{\text{nat.}} \supseteq \mathbb{R}[U] = R$, so $V^{\text{nat.}} = R$. Since $J^{\text{nat.}}$ is an ideal of the ring $V^{\text{nat.}} = R$, $J^{\text{nat.}} = 0$ follows. Now the mapping

$$x \to \operatorname{Inf}_{\mathbb{R}}(q \text{ rational} \mid xx^* \leqslant q^2\}$$

turns R into a normed ring. Thus R is a normed division algebra. By the well-known Gel'fand–Mazur theorem R would be of dimension at most 4, which completes the proof.

If the span $\mathbb{R}[U]$ of the unitary group U over the real $\mathbb{R}$ is a simple ring, this is no longer under the Gel'fand–Mazur domain of validity and we cannot conclude that R must be, again, of dimension 1 or 4. All we can say in that case is that $A = \mathbb{R}[U]$ is a simple algebra without divisors of zero, which is spanned by unitaries.

If, however, the division ring R is c-orderable the situation is quite different for we can show the

THEOREM 10. (1) *If R is any non-commutative c-orderable division ring of dimension not* 4, *then the span of the unitary group of R over any subfield F of naturally bounded central elements cannot be a simple ring.* (2) *Also, the span of the pseudo unitary group of R over any such subfield F of Z cannot be a simple ring.*

Proof. (1) When $F \subseteq Z_b$, $A = F[U] \subseteq V$ follows. Let I be the ideal of A generated the commutators $sx - xs$, where $s = s^*$ and x range over A. By Theorem 4, $I \subseteq J$. If now A were a simple ring, we would get $I = 0$ or $I = A$. In the former case one can easily conclude that R is of dimension at most 4. In the latter case this would give $A \subseteq J$ and consequently $1 \in J$, which is an absurdity.

(2) Let $\tilde{U}$ be the set of elements of the form

$$u = x^* x^{-1} \cdots t^* t^{-1} \qquad (x, \ldots, t \in R),$$

or *pseudo unitary* elements. It can be easily checked that $\tilde{U}$ is a multiplicative group, which is preserved under the involution. By an observation due to Holland, the ring $A = F[\tilde{U}]$ is such that every one-sided ideal is two-sided. For if L is a right ideal of A and $x \in L$, then $xx^{-1}x^* \in L$, that is, $x^* \in L$. If now A is a simple ring, it follows that A is a division subring. By an observation due, again, to Holland, we have $aba^{-1}b^{-1} \in \tilde{U}$ for all $a, b \in R$, $a, b \neq 0$. Therefore A is preserved under conjugation. By the famous Brauer–Cartan–Hua theorem either $A \subseteq Z$ or $A = R$. In the former case, we can easily conclude that R is at most 4-dimensional, which is contrary to the hypothesis. In the latter case, we get that $V \supseteq A = R$. From this $V = R$, and $J = 0$ follows. But, this can happen only when R is at most 4-dimensional.

In the course of the preceding proof the following corollary was almost shown.

COROLLARY 1. *Let R be any c-orderable division ring whose center Z is totally archimedean (for example, Z is an algebraic number field or $Z = \mathbb{R}$ or $\mathbb{C}$). The span A of the pseudo unitary group U of R over the center has the following properties.*

(i) *Every one-sided ideal of A is 2-sided.*

(ii) *A is preserved under conjugation.*

(iii) *Unless R is at most 4-dimensional neither $A \subseteq Z$ nor can A be a simple ring.*

(iv) *The center of A is precisely Z.*

(v) *A enlarges to a proper *-valuation subring.*

Property (iv) of Corollary 1 follows easily from the fact that the center of A is preserved under conjugation. In the case where the division ring R contains in its center the reals, we can assert that A contains an ideal J_A with A/J_A one of the real finite dimensional algebras. For then V/J is a real division algebra whose symmetrics are reals. This can be shown as follows. Taking $s = s^*$ any positive symmetric unit, if we define $\|s\| = \mathrm{Inf}_{\mathbb{R}}\{q(\text{rational}) \mid s \leqslant q\}$ then s and $\|s\|$ have the same rational majorants, so that $s \equiv \|s\| \pmod J$ (Theorem 5). Since A/J is a real subalgebra of V/J it follows that A/J is a division subalgebra. Thus $A/J \cap A \approx A + J/J = A/J$ is a division algebra. If, further, R contains in its center the complexes then $V/J = \mathbb{C} = A/J$. Here the commutator ideal of V is contained in J so that A has a proper commutator ideal. This is the

COROLLARY 2. *If, further, the center of R is $\mathbb{C}$ then the commutator ideal of A is a proper ideal and A has some maximal ideal J_A with $A/J_A \approx \mathbb{C}$.*

Remark 10. Under the assumptions of Corollary 2, we have

$$Z[U] + J = Z[\tilde{U}] + J = V^{\text{nat.}} + J = V,$$

where $V^{\text{nat.}}$ ($\supseteq Z[\tilde{U}] \supseteq Z[U]$) *is the subring of naturally bounded elements.*

4.3. *Examples*

4.3.1. *Algebraic Case*

We wish to determine all algebraic division rings R possessing a joint Baer and c-ordering. Here is a construction of the non-commutative case, which turns out to be the most general one.

Given the q-ordered field F and the quadratic extension $M = F[\tau]$ of F of the form $-\tau^2 > 0$ in F, we shall associate to the pair $(F; \tau)$ the 4-dimensional division ring $R = R(F; \tau, \geqslant)$ as follows. We let R be all the 2×2 matrices over M of the form

$$\begin{bmatrix} a & b \\ \tau^2 \bar{b} & \bar{a} \end{bmatrix},$$

where $x \mapsto \bar{x}$ is the Galois F-automorphism of M induced by τ. This is a division subring of $M_{2\times 2}$ with the involution (symplectic and conjugate–transpose)

$$\begin{bmatrix} a & b \\ \tau^2 \bar{b} & \bar{a} \end{bmatrix}^* = \begin{bmatrix} \bar{a} & -b \\ -\tau^2 \bar{b} & a \end{bmatrix}.$$

Here R has for center $Z = F = \{x \in R \mid x = x^*\}$. If $M = \{a \in F \mid a > 0\}$ then M induces a c-ordering and a Baer ordering on R, for all symmetrics are scalars in F.

Conversely if $R \neq Z$ is algebraic with some joint Baer and c-ordering we know that all symmetrics are central. Such a division ring R has an involution of the first kind necessary. Thus the center Z is q-ordered by the restriction of the c-ordering to Z. If now $\tau = -\tau^* \neq 0$ is any skew element of R then $-\tau^* = \tau\tau^* \in Z$. By construction, $-\tau^2$ is a positive element of the q-ordered field Z. Choosing another skew σ anti-commutating with τ and using a reduced matrix argument to the basis $1, \tau, \sigma, \tau\sigma$ of R over Z, one gets that $R \approx R(F; \tau)$.

If we stick to a merely c-orderable division ring R we can turn, in counterpart, to the case where the $*$-center $(Z, *)$ of R is totally archimedean. We know from Corollary 2 to Theorem 8 that R has all its symmetrics central. For $R \neq Z$, this gives that $Z = (Z, *)$ is totally archimedean. In this case, any q-ordering on Z is a field ordering. Thus $R \approx R(F; \tau; \geqslant)$, where $\geqslant$ is a field ordering on $F = Z$, with F totally archemedean, and $-\tau^2 > 0$ in F. Conversely, if $R = R(F; \tau; \geqslant)$, where F, $\geqslant$, and τ are as before, then R has a totally archimedean $*$-center $(Z, *) = Z = F$, and R is c-ordered. We have show the

THEOREM 11. *The only algebraic c-orderable division rings $R \neq Z$ which possess a joint Baer and c-ordering (resp. which possess a totally archimedean $*$-center) are the division rings of the form $R = R(F; \tau; \geqslant)$, where $(F, \geqslant)$ is a q-ordered field (resp. $(F, \geqslant)$ is an ordered field, which is totally archimedean) and $-\tau^2$ is a positive element of F.*

4.3.2. *Weyl Division Algebras*

Start off any field with involution F, which is naturally orderable. Denote by D the Weyl algebra over F, $D = F(x, y]/(xy - yx - 1)$, relative to the involution, making x symmetric and y skew. As is well known, D is a simple noetherian ring with center F. We let R be the division ring of quotients of D together with the extended involution. We claim that R possesses some joint Baer and c-ordering. Here is a construction of such an ordering.

View D as the polynomial ring $F[y][x]$ over the commutative domain $F[y]$ subjected to the rule

$$\phi \in F[y], x^i\phi = \phi x^i + i\frac{d\phi}{dy}x^{i-1} + \cdots + \binom{i}{j}\frac{d^j\phi}{d\phi^j}x^{i-j} + \cdots + \frac{d^i\phi}{dy^i},$$

where $d^k\phi/dy^k$ is the higher derivative of order k of the polynomial $\phi = \phi(y)$. Using this rule one can show that D possesses a mapping $N: D \to F[y]$ such that $N(p) = 0 \Leftrightarrow p = 0$, $N(pg) = N(p)\,N(g)$, $N(p^*) = (N(p))^*$, $N(c) = c$ for each $c \in F[y]$, and $N(\sum_{j=1}^{k} p_i) = \sum_{i_j} N(p_{ij})$, where $\{i_j\}_j$ is a subset (possibly empty) of indices. For example, define $N(p) = 0$ for $p = 0$, and $N(p) = a_n(y)$, where $0 \neq p = \sum_{i=0}^{n} a_i(y) \in D$.

Clearly $F[y]$ has a naturally orderable field of quotients $F(y)$, which possesses some multiplicative c-ordering. For example, take a field ordering of the *-center $(F(y), {}^*)$ of $F(y)$ and set $c \in F(y)$, $c > 0 \Leftrightarrow c \in (F(y), {}^*)$, $c > 0$. Define $p \in D$ to be positive if and only if $N(p) \in F[y]$ and $N(p) > 0$. The set $M = \{p \in D \mid p > 0\}$ is an additive and multiplicative subset defining a joint Baer and c-ordering on D. We claim that this additive an multiplicative set M has the Ore property: Given p, $g \in M$ there are $p_1, g_1 \in M$ with $p_1 p = g_1 g$. For since D has the Ore property there are p_2 and g_2 with $p_2 p = g_2 g$. Then $(p_2^* p_2)p = (p_2^* g_2)g$. Now $p_2^* p_2$ and $p \in M$ so that $(p_2^* g_2)g = (p_2^* p_2)p > 0$. Also $g > 0$. Thus $N((p_2^* g_2)g) = N(p_2^* g_2)\,N(g) > 0$ and $N(g) > 0$. By construction of the ground c-ordering on $F[y]$, we get $N(p_2^* g_2) > 0$. Therefore $p_2^* g_2 \in M$. This shows that M satisfies the Ore condition as a cone. This cone M can be inversed in the division ring of quotients R of D; that is, $\bar{M} = \{pg^{-1} \mid p, g \in M\}$ is also an additive and multiplicative cone in R. $\bar{M}$ contains norms of R. For let $x - b^{-1}a \in R$. Then

$$ss^* = b^{-1}aa^*(b^*)^{-1}$$

$$ss^* = (b^*b)^{-1}(b^*a)(b^*a)^*(b^*b)^{-1} \in \bar{M}.$$

Therefore $\bar{M}$ induces a joint Baer and c-ordering on the division ring R, which is multiplicative.

If instead of the chosen involution we had taken the exchange involution $x \to y = x^*$, the same conclusion would be true for R. It suffices to observe that $D = F[x', y']/(x'y' - y'x' - 1)$, where $x' = \frac{1}{2}(x + y)$, $y' = y - x$. Here the involution $x \to y$ does make x' symmetric and y' a skew element.

Taking the field F to be a totally archimedean *-center $(F, *)$, we know from Handelman's result that, in the case $F = \mathbb{R}$ or $\mathbb{C}$, $R = R(F)$ can not be spanned by its unitaries over F. From our earlier discussion (see 4.2, Theorem 10), we know, in fact, that $A = F[U]$ or even $A = F[\bar{U}]$ are not simple rings. In the case $F = \mathbb{C}$, $A + J = V$ follows with $A/A \cap J \approx \mathbb{C}$. The same conclusions can be derived for the Weyl algebra itself.

4.3.3. *Holland's Examples*

The question of whether any naturally orderable division ring R is c-orderable can be answered in the negative as Holland constructed a 9-dimensional naturally orderable division ring R, which is algebraic over the rationals and by [5, Theorem 8].

In [5], Holland gave an example of infinite dimensional division ring R with pre-selected *-center Z_0, which satisfies the square root axiom, and R is Baer orderable provided Z_0 is formally real. This example turns out to c-orderable, and this is checked by the author in a separate paper.

In [6], Holland gave examples of odd-dimensional division rings R, which are Baer orderable. It would be interesting to test for the c-orderability of those division rings, To conclude, here are some other questions.

QUESTION 1. If R is a c-orderable division ring must R possess a joint Baer and c-ordering? a separate ordering?

QUESTION 2. If R is any c-orderable division ring must all algebraic symmetrics be central elements?

QUESTION 3. If R is not a quaternion ring and if R has no algebraic symmetrics (other than the central ones) must R be a purely transcendental extension of its center Z?

QUESTION 4. For R any naturally orderable division ring, when is the span of the unitary (pseudo unitary) group of R over the center a simple ring? a local ring? Must the latter spans be always equal?

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